AP CALCULUS AB — Homework 0210 SOLUTIONS
DR. PAUL L. BAILEY — Monday, February 10, 2025

Problem 1 (Thomas §4.1 # 18)**.** Let $f : [-3, 1] \to \mathbb{R}$ be given by $f(x) = 4 - x^2$. Find the absolute maximum and minimum values of f. Graph the function, identifying the points of the graph where the absolute extrema occur, and include their coordinates.

Solution. First find the critical points. Compute $f'(x) = -2x$. Solve $f'(x) = 0$ to get $x = 0$. This is the only critical point. Plug the critical points and the endpoints into f:

- $f(-3) = -5$
- $f(1) = 3$
- $f(0) = 4$

The absolute min is at $x = -3$. The absolute min value is $f(-3) = -5$.
The absolute max is at $x = 0$. The absolute max value is $f(0) = 4$. □

Problem 2 (Thomas §4.1 # 23)**.** Let $f : [-2, 1] \to \mathbb{R}$ be given by $f(x) = \sqrt{4 - x^2}$. Find the absolute maximum and minimum values of f. Graph the function, identifying the points of the graph where the absolute extrema occur, and include their coordinates.

Solution. First find the critical points. Compute $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$. Solve $f'(x) = 0$ to get $x = 0$. This is the only critical point. Plug the critical points and the endpoints into f:

- $f(-2) = 0$
- $f(1) = \sqrt{3}$
- $f(0) = 2$

The absolute min is at $x = -2$. The absolute min value is $f(-2) = 0$.
The absolute max is at $x = 0$. The absolute max value is $f(0) = 2$. □

Problem 3 (Thomas §4.1 # 43)**.** Let $f(x) = \dfrac{x}{x^2 + 1}$. Find the extreme values of f and where they occur. Find the domain and range of f.

Solution. We see that $\operatorname{dom}(f) = \mathbb{R}$.

Take the derivative to find the extreme values. Compute $f'(x) = \dfrac{1 - x^2}{(1 + x^2)^2}$. Solve $f'(x) = 0$ to get $x = \pm 1$. Plug in these critical points to get

- $f(1) = \dfrac{1}{2}$
- $f(-1) = -\dfrac{1}{2}$

Thus $\operatorname{range}(f) = [-\frac{1}{2}, \frac{1}{2}]$. □

Problem 4 (Thomas §4.1 # 48)**.** Let $f(x) = x^2\sqrt{3-x}$. Find all critical points of f. Determine the local extreme values of f.

Solution. The domain of f is $(-\infty, 3]$.

Take the derivative to find the critical points. Compute $f'(x) = \dfrac{12x - 5x^2}{2\sqrt{3-x}}$. Solve $f'(x) = 0$ to get $x = 0$ or $x = \frac{12}{5}$. Plug the critical points and the endpoints into f to get

- $f(3) = 0$
- $f(\frac{12}{5}) = \frac{144}{25} \cdot \sqrt{\frac{3}{5}}$
- $f(0) = 0$

The absolute min is at $x = 0$. The absolute min value is $f(0) = 0$.
The absolute max is at $x = \dfrac{12}{5}$. The absolute max value is $f(\frac{12}{5}) = \frac{144}{25} \cdot \sqrt{\frac{3}{5}}$. □

Problem 5 (Thomas §2.6 # 35)**.** Define $g(3)$ in a way that extends $g(x) = (x^2-9)/(x-3)$ to be continuous at $x = 3$.

Solution. The domain of g is $\mathbb{R} \setminus \{3\}$. On this domain, we have $g(x) = \dfrac{(x+3)(x-3)}{x-3} = x + 3$. Now $(x+3)\mid_3 = 6$, so if we define $g(3) = 6$, then g becomes a linear function. More accurately, we define a new function

$$\widehat{g} : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad \widehat{g}(x) = \begin{cases} g(x) & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

This is a continuous extension of g. □

Problem 6 (Thomas §3.8 # 9)**.** Find the linearization of $f(x) = \sqrt[3]{x}$ at $a = 1$, and use it to approximate $f(1.3)$.

Solution. Compute $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, and $f'(1) = \dfrac{1}{3}$. Also, $f(1) = 1$. Thus

$$L(x) = \frac{1}{3}(x-1) + 1.$$

So $L(1.3) = 1.1$. □

Problem 7 (APCalcAB.1969.MC.8)**.** Let $p(x) = (x+2)(x+k)$. Suppose that the remainder is 12 when $p(x)$ is divided by $x - 1$. Find k.

Solution. Let r be the remainder. The remainder theorem says that $r = p(1) = 3(1+k) = 3 + 3k$. It is given that $r = 12$, so $3 + 3k = 12$, so $k = 3$. □

Problem 8 (APCalcAB.1969.MC.18)**.** Let $f(x) = 2 + |x - 3|$. Find $f'(x)$ and $f'(3)$.

Solution. Since f is a piecewise defined function, so is its derivative:

$$f'(x) = \begin{cases} -1 & \text{if } x < 3 \\ 1 & \text{if } x > 3 \end{cases}$$

But f is not differentiable at $x = 3$, so $f'(3)$ does not exist. □

Problem 9. Compute

$$\frac{d^{999}}{dx^{999}} \sin x.$$

Solution. Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and $f^{(4)}(x) = \sin x$. This pattern continues:

$$f^{(n)}(x) = \begin{cases} \sin x & \text{if} \quad n \equiv 0 \pmod 4 \\ \cos x & \text{if} \quad n \equiv 1 \pmod 4 \\ -\sin x & \text{if} \quad n \equiv 2 \pmod 4 \\ -\cos x & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}$$

Since $999 \equiv 3 \pmod 4$, we have $f^{(999)} = -\cos x$. □

Problem 10. Let H denote the set of all henhouses in the United States, and let C denote the set of all chickens that live in one henhouse. Let

$$f : C \to H \quad \text{be given by} \quad f(\texttt{chicken}) = \texttt{henhouse in which chicken lives}.$$

Let

$$g : H \to \mathbb{R} \quad \text{be given by} \quad g(\texttt{henhouse}) = \texttt{area in square footage of the henhouse}.$$

Let

$$h : C \to \mathbb{R} \quad \text{be given by} \quad h(c) = \frac{g \circ f(c)}{|f^{-1}(f(c))|}.$$

Suppose $c \in C$. Describe $h(c)$.

Solution. We see that $f(c)$ is the henhouse of chicken c, so $g \circ f(c) = g(f(c))$ is the square footage of that henhouse. Also $f^{-1}(f(c))$ is the preimage of $f(c)$, so it is all of the chickens which live in henhouse $f(c)$. Thus $|f^{-1}(f(c))|$ is the number of chickens which live in that henhouse, and

$$\frac{g \circ f(c)}{|f^{-1}(f(c))|} = \frac{\texttt{Square footage}}{\texttt{Number of chickens}} = \texttt{Square footage per chicken}.$$

That is, this represents the amount of space that chicken c has to herself. □